Novelty Journals

ISSN 2394-9651

International Journal of Novel Research in Physics Chemistry & Mathematics
Vol. 6, Issue 2, pp: (15-22), Month: May - August 2019, Available at: www.noveltyjournals.com

ACTION OF THE DIHEDRAL GROUP D_5 ON X= {1,2,3,4,5}

CHEPKORIR BETTY

MATHEMATICS DEPARTMENT, SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, KENYATTA UNIVERSITY

Abstract: **Let G be a group acting transitively on a set X, where $|X|$= n. Higman (1964) showed that if G is of rank 3 and degree $n=k^2+1$, where k is the length of of a $G_x - orbit$, $x \in X$ then $n=5$,10, and 50. He further show that such groups of degrees 5, 10 and 50 exist namely**

i). The dihedral group D_5 of order 10 has such a representation of degree 5,

ii). The alternating group A_5 and the symmetric group S_5 on 5 points acting on the set of unordered pairs of distinct points provide examples of degree 10,

iii). $U_3(5)$ has rank 3 representation of degree 50 of this kind, as does the group obtained from it by adjoining the field automorphism. Degree 3250 remains undecided.

In this project we shed more light on the groups in parts (i) and (ii). We study the subdegrees, suborbital graphs and intersection matrices corresponding to the representations of these groups. Properties of the suborbital graphs and intersection matrices associated with these representations will also be investigated.

1. PROBLEM STATEMENT

In this project we are investigating the properties of rank 3 groups of degree K^2 +1.We intend to shed more light on the following groups:

i). The dihedral group D_5 of order 10 with a representation of degree 5,

ii). The alternating group A_5 and the symmetric group S_5 on 5 points acting on the set of unordered pairs of distinct points provide examples of degree 10,

We deal with the following problems:

i). Finding the subdegrees of these three groups.

ii). Finding the suborbits and constructing suborbital graphs associated with the action and discussing the properties of these graphs.

iii). Finding the intersection numbers associated with each non- trivial suborbit .

iv). Finding the intersection matrix associated with each non –trivial suborbit and discussing the properties of these matrices.

2. OBJECTIVES

Our main aim is to study the ranks, suborbits, subdegrees , suborbital graphs ,intersection matrices corresponding to the action of the symmetric group S_5 and the alternating group A_5 on the set of unordered pairs from the set X={1,2,3,4,5}.we also study the action of the dihedral group D_5 on the set X ={ 1,2,3,4,5 }.

International Journal of Novel Research in Physics Chemistry & Mathematics
Vol. 6, Issue 2, pp: (15-22), Month: May - August 2019, Available at: www.noveltyjournals.com

We intend to construct the suborbital graphs associated with the action and discuss their properties. And also compute the intersection numbers to come up with the intersection matrices.

3. INTRODUCTION

We investigate some properties of the action of D_5 on X={ 1,2,3,4,5 }. It is presented in three sections:

Section 1 deals with the subdegrees of D_5 acting on X

Section 2 deals with the suborbits of D_5 and the corresponding suborbital graphs.

Finally in section 3 , we find the intersection numbers and the intersection matrices associated with each non –trivial suborbit.

Subdegrees of G=D_5 on X={ 1,2,3,4,5 }

Let G act on the set X={1,2,3,4,5}.

Lemma 1

G acts transitively on X={1,2,3,4,5}

Proof

Using the orbit- stabilizer theorem (Theorem 1.1.3.10) , we need to show that the length of the orbit of a point say 1 is five same as the number of points in X .This implies that that the action of G on X has only one orbit .

Taking 1 in *X*, $Stab_G(1) = \{1, (34)(25)\}$

Hence $|Stab_G(1)| = 2$

Applying the orbit –stabilizer theorem, we get

$$Orbit_G(1) = |G : Stab_G(1)|$$

$$= \frac{|G|}{|Stab_G(1)|} = \frac{10}{2} = 5$$

Thus the orbit of $1 \in X$ is the whole of *X* .Therefore *G* acts transitively on *X*.

Lemma 2

The number of orbits of G_1 on X is 3.

Proof

To prove this we apply the Cauchy – Frobenius lemma (Theorem 1.1.3.8) .We show that the number of orbits of the stabilizer of 1 in D_5 is 3.

$$Stab_G(1) = \{1, (25)(34)\} = G_1$$

Hence

$$Orb_{G_1}(1) = \{1\}$$

$$Orb_{G_1}(2) = \{2, 5\}$$

$$Orb_{G_1}(3) = \{3, 4\}$$

Thus the number of orbits of G_1 on X is 3 , implying that the rank of G on X is 3.

The three orbits of G_1 acting on X determined above are :

$\Delta_0 = Orb_{G_1}(1) = \{1\}$. The trivial orbit

$\Delta_1 = Orb_{G_1}(2) = \{2,5\}$, The orbit containing 2 and 5

$\Delta_2 = Orb_{G_1}(3) = \{3,4\}$, The orbit containing 3 and 4

Therefore, the subdegrees of D_5 on X are 1 , 2 and 2 .

Suborbital graphs of $G = D_5$

From the previous section , the suborbits of G are :

$\Delta_0 = Orb_{G_1}(1) = \{1\}$.

$\Delta_1 = Orb_{G_1}(2) = \{2,5\}$,

$\Delta_2 = Orb_{G_1}(3) = \{3,4\}$,

Now

$X \times X$ = {(1 1) , (12) , (13) , (14) , (15) , (21) , (22) , (23), (24), (25) , (31) (32), (33) , (34) , (35) , (41) , (42) , (43), (44), (45) , (51) (52) , (53), (54) , (55)} .

By lemma 1.1.5.4 , we find that the suborbitals corresponding to the suborbits Δ_0, Δ_1 and Δ_2 are:

O_0 (1,1) = { (1,1) , (2,2) , (3,3) , (4,4) , (5,5) }

O_1(1,2) = {(1,2) , (2,3), (3,4) , (4,5) , (5,1) , (1,5) , (3,2), (5,4) , (2,1) ,(4,3)}

O_2(1,3) = {(1,3) , (2,4), (3,5) , (4,1) , (5,2) , (1,4) , (3,1) , (5,3), (2,5) , (4,2)}

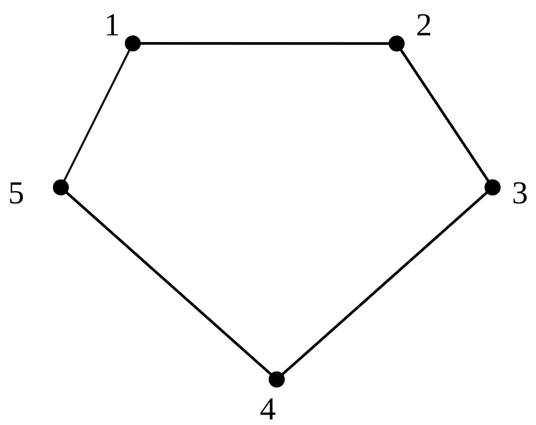
From these suborbitals, we find the suborbital graphs. The suborbital graph corresponding to O_0 is the null graph.

We now consider the suborbital graphs corresponding to the suborbitals O_1 and O_2 respectively.

Since the order of D_5 is 10 which is even, by Theorem 1.1.5.6 [Wielandt , 1964 ,section 16.5] Δ_1 and Δ_2 are self-paired . Hence their corresponding suborbital graphs are undirected.

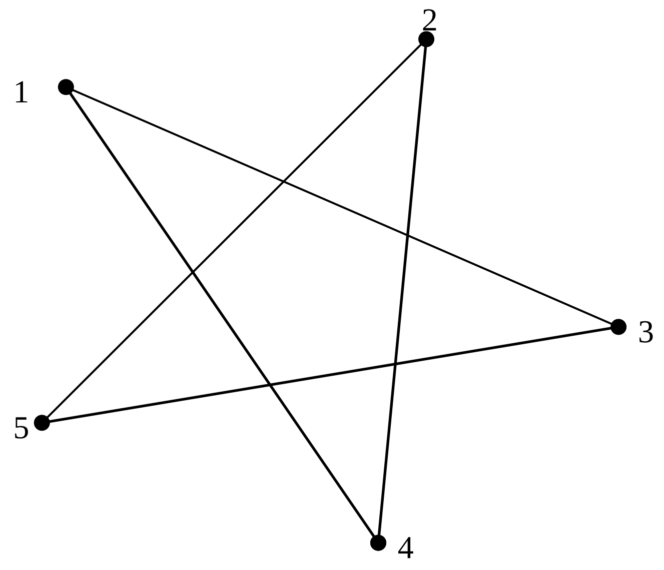
We then construct the suborbital graphs Γ_1 and Γ_2 in figure 5.2.1 and Figure 5.2.2 respectively.

The suborbital graph corresponding to the suborbit Δ_1 of G on X



Γ_1 is regular of degree degree 2. It is also connected

The suborbital graph Γ_2 corresponding to the Δ_2 of *G* on *X*..



Γ_2 is regular of degree 2 and is connected.

3. Intersection matrices associated with the action of G=D_5 on *X*

In this section we compute the intersection numbers and the corresponding intersection matrices associated with each non-trivial suborbit Δ_1 and Δ_2.

By Definition 1.1.6.1, given an arrangement of the G_a-orbits , the G_b–orbits are arranged such that if b$\in$X and $g(a) = b$ then,

$$g\left(\Delta_l\left(a\right)\right) = \Delta_i\left(g\left(b\right)\right) = \Delta_l\left(b\right)$$

Intersection matrix corresponding to $\Delta_1(1)$

From a general discussion of intersection numbers and intersection matrices in section 1.1.6.

Now taking $a = 1$ in X and G_1-orbits arranged as follows,

$\Delta_0(1) = \{1\}$.

$\Delta_1(1) = \{2,5\}$,

$\Delta_2(1) = \{3,4\}$,

We arrange the G_b- orbits as follows :

$\Delta_0(2) = \{2\}$.

$\Delta_1(2) = \{1,3\}$,

$\Delta_2(2) = \{4,5\}$,

International Journal of Novel Research in Physics Chemistry & Mathematics
Vol. 6, Issue 2, pp: (15-22), Month: May - August 2019, Available at: www.noveltyjournals.com

$\Delta_0(3) = \{3\}$.

$\Delta_1(3) = \{1,5\}$,

$\Delta_2(3) = \{2,4\}$,

From definition 1.1.6.1, the intersection numbers relative to the suborbit $\Delta_1(1)$ are defined by

$$\mu_{ij}^{(l)} = \left|\Delta_l(b) \cap \Delta_i(1)\right|, \qquad b \in \Delta_j(1),$$

Hence we find the intersection numbers relative to $\Delta_1(1)$ as follows

$$\mu_{00}^{(1)} = \left|\Delta_1(1) \cap \Delta_0(1)\right| = 0$$

$$\mu_{10}^{(1)} = \left|\Delta_1(1) \cap \Delta_1(1)\right| = 2$$

$$\mu_{20}^{(1)} = \left|\Delta_1(1) \cap \Delta_2(1)\right| = 0$$

$$\mu_{01}^{(1)} = \left|\Delta_1(2) \cap \Delta_0(1)\right| = 1$$

$$\mu_{11}^{(1)} = \left|\Delta_1(2) \cap \Delta_1(1)\right| = 0$$

$$\mu_{21}^{(1)} = \left|\Delta_1(2) \cap \Delta_2(1)\right| = 1$$

$$\mu_{02}^{(1)} = \left|\Delta_1(3) \cap \Delta_0(1)\right| = 1$$

$$\mu_{12}^{(1)} = \left|\Delta_1(3) \cap \Delta_1(1)\right| = 1$$

$$\mu_{22}^{(1)} = \left|\Delta_1(3) \cap \Delta_2(1)\right| = 0$$

By definition 1.1.6.2 the intersection matrix $M_1 = \left(\mu_{ij}^{(1)}\right)_{i,j}$, associated with $\Delta_1\{1,2\}$ where $\mu_{ij}^{(1)}$ are the intersection numbers relative to $\Delta_1(1)$ is obtained as follows;

$$M_1 = \begin{bmatrix} \mu_{00}^{(1)} & \mu_{01}^{(1)} & \mu_{02}^{(1)} \\ \mu_{10}^{(1)} & \mu_{11}^{(1)} & \mu_{12}^{(1)} \\ \mu_{20}^{(1)} & \mu_{21}^{(1)} & \mu_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Intersection matrix corresponding to $\Delta_2(1)$

From definition 1.1.6.1, the intersection numbers relative to the suborbit $\Delta_2(1)$ are defined by

$$\mu_{ij}^{(2)} = \left|\Delta_2(b) \cap \Delta_i(1)\right|, \qquad b \in \Delta_j(1),$$

We therefore find the intersection numbers relative to $\Delta_2(1)$ as follows

ISSN 2394-9651

International Journal of Novel Research in Physics Chemistry & Mathematics
Vol. 6, Issue 2, pp: (15-22), Month: May - August 2019, Available at: www.noveltyjournals.com

$$\mu_{00}^{(2)} = \left|\Delta_2(1)\,\mathrm{I}\,\Delta_0(1)\right| = 0$$

$$\mu_{10}^{(2)} = \left|\Delta_2(1)\,\mathrm{I}\,\Delta_1(1)\right| = 0$$

$$\mu_{20}^{(2)} = \left|\Delta_2(1)\,\mathrm{I}\,\Delta_2(1)\right| = 2$$

$$\mu_{01}^{(2)} = \left|\Delta_2(2)\,\mathrm{I}\,\Delta_0(1)\right| = 0$$

$$\mu_{11}^{(2)} = \left|\Delta_2(2)\,\mathrm{I}\,\Delta_1(1)\right| = 1$$

$$\mu_{21}^{(2)} = \left|\Delta_2(2)\,\mathrm{I}\,\Delta_2(1)\right| = 1$$

$$\mu_{02}^{(2)} = \left|\Delta_2(3)\,\mathrm{I}\,\Delta_0(1)\right| = 0$$

$$\mu_{12}^{(2)} = \left|\Delta_2(3)\,\mathrm{I}\,\Delta_1(1)\right| = 1$$

$$\mu_{22}^{(2)} = \left|\Delta_2(3)\,\mathrm{I}\,\Delta_2(1)\right| = 1$$

By definition 1.1.6.2 the intersection matrix $M_2 = \left(\mu_{ij}^{(2)}\right)_{i,j}$, associated with $\Delta_2(1)$ where $\mu_{ij}^{(2)}$ are the intersection numbers relative to $\Delta_2(1)$ is obtained as follows;

$$M_2 = \begin{bmatrix} \mu_{00}^{(2)} & \mu_{01}^{(2)} & \mu_{02}^{(2)} \\ \mu_{10}^{(2)} & \mu_{11}^{(2)} & \mu_{12}^{(2)} \\ \mu_{20}^{(2)} & \mu_{21}^{(2)} & \mu_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Properties of the intersection matrices associated with $\Delta_2(1)$ and $\Delta_2(1)$

By computation of the intersection matrices we are able to come up with the following properties.

i). The column sum of the intersection matrix associated with Δ_i is equal to the degree (valency) of the suborbital graph corresponding to the same suborbit Δ_i , which is also the length of the suborbit.

We can see that the column sum of M_1 is 2 equal to the degree of Γ_1 .Also the column sum of M_2 is 2 equal to the degree of Γ_2 .

ii). M_1 and M_2 are square matrices .

iii). The order of M_1 and M_2 is 3×3 since the rank of D_5 is 3.

4. CONCLUSION

In this project we investigated some properties of the action of D_5 on X = { 1,2,3,4,5 } , we showed that D_5 acts transitively on X. We found the rank of D_5 when it acts on X to be 3 , same as that obtained by Higman (1964) .And that the subdegrees of D_5 are 1 , 2 and 2 . We also constructed the suborbital graphs and found out that suborbital graphs Γ_1

and Γ_2 corresponding to the non-trivial suborbits of D_5 are regular and undirected. We computed the intersection numbers and intersection matrices associated with each non – trivial suborbit Δ_1 and Δ_2. We found out that the intersection matrices M_1 and M_2 are square matrices and that the column sum of M1 is 2 equal to the degree of Γ_1 and the column sum of M_2 is also 2 equal to the degree of Γ_2 .

REFERENCES

[1] Akbas, M. 2001. Suborbital graphs for modular group, *Bulletin of the London mathematical society* 33:647-652.

[2] Burnside, W. 1911. *Theory of groups of finite order*, Cambridge University Press, Cambridge (Dover reprint 1955).

[3] Bon, J. V. and Cohen, A.M. 1989. Linear groups and distance-transitive groups, *European Journal of combinatories* 10:399-411.

[4] Cameron, P. J. 1972. Permutation groups with multiply transitive suborbits I. Proc. London Math. *Soc*. 23 (3): 427 – 440.

[5] Cameron, P. J. 1974. *Permutation groups with multiply transitive suborbits II*. Bull. London Math. *Soc*. 6: 136 - 140.

[6] Cameron, P. J. 1978. *Orbit of permutation groups on Unordered sets.* J. London. Math. *Soc*. 17 (2): 410- 414.

[7] Chartrand, G. 1993. *Applied and algorithmic graph theory*. International series in pure and applied Mathematics: 238-240.

[8] Coxeter, H. S. M. 1986. My graph, *proceedings of London mathematical society* 46:117-136.

[9] Faradžev, I. A. and Ivanov, A. A. 1990. *Distance-transitive representations of group G with PSL (2q)<-G< PTL (2,q), European journal of combinatories* 11:347-356.

[10] Harary, F. 1969. *Graph Theory*. Addison – Wesley Publishing Company, New York.

[11] Higman, D. G. 1964. *Finite permutation groups of rank 3.* Math Zeitschriff 86: 145 – 156.

[12] Higman, D. G. 1970. *Characterization of families of rank 3 permutation groups by subdegrees*. I. Arch. Math 21: 151-156.

[13] Jones, G. A.; Singerman, D. and Wicks, K., 1991. *Generalized Farey graphs in groups*, St. Andrews 1989, Eds. C. Campbell and E.F. Robertson, London mathematical society lecture notes series 160, Cambridge University, Cambridge 316 – 338.

[14] Kamuti, I. N. 1992. *Combinatorial formulas, invariants and structures associated with primitive permutation representations of PSL (2, q) and PGL* (2, *q*). Ph. D. Thesis, Southampton University, U.K.

[15] Kamuti, I. N. 2006. *Subdegrees of primitive permutation representation of PGL (2,q)*, East African journal of physical sciences 7(1/2):25-41.

[16] Kangogo, M. 2008. *Some properties of symmetric group S6 and associated combinatorial formulas and structures*, M.SC. Project; Kenyatta, University, Kenya.

[17] Krishnamurthy, V. 1985. *Combinatorics, theory and applications*, Affliated East West Press Private Limited, New Delhi.

[18] Neumann, P. M. 1977. *Finite permutation groups edge-coloured graphs and matrices* edited by M. P. J. Curran, Academic Press, London.

[19] Petersen, J. 1898. Sur le. Theore'me de Tait Intermed Math 5:225-227.

[20] Rose, J. S. 1978. *A Course on group theory*. Cambridge University Press, Cambridge.

[21] Rosen, K. H. 1999. *Handbook of Discrete and combinational Mathematics*. CRC Press , New Jersey.

International Journal of Novel Research in Physics Chemistry & Mathematics
Vol. 6, Issue 2, pp: (15-22), Month: May - August 2019, Available at: www.noveltyjournals.com

[22] Rotman, J. J. 1973. *The theory of groups*: *An introduction*. Allyn and Bacon, Inc. Boston, U.S.A.

[23] Rotich, K . S. 2008. *Some properties of the symmetric group S_7 acting on ordered and unordered Pairs and the associated combinatorial structures* , M.SC. Project , Kenyatta University ,Kenya.

[24] Galois, C. 1830. *Self Conjugate sub-groups; simple and composite groups.* Cambridge University Press Cambridge.

[25] Jordan, C. 1870. *Traite des substitutions et des Equations Algebriques*. Gauthier – Villas, Paris.

[26] Sims, C. C. 1967. *Graphs and finite permutation groups.* Math. Zeitschrift 95: 76 – 86.

[27] Wielandt, H. 1964. *Finite Permutation Groups*. Academic Press, New York and London.